CFT & gravity, week 2

Exercise 1.2.7 Coupling redefinitions and the β-function

$$\beta(g) = \frac{dg}{d\log\Lambda}, \quad \beta(0)=0 \quad \text{fixed point}$$

$\tilde{g}(g)$ redefinition such that $\tilde{g}(0)=0$

$$\tilde{\beta}(\tilde{g}) = \frac{d\tilde{g}}{d\log\Lambda}$$

Perturbation theory:

$$\tilde{g}(g) = a_1 g + a_2 g^2 + \dots$$

$$\beta(g) = b_1 g + b_2 g^2 + \dots \qquad b_1 \neq 0$$

We want to show that a_i can be chosen so that

$$\tilde{\beta}(\tilde{g}) = c\,\tilde{g} \qquad (*)$$

$$\tilde{\beta}(\tilde{g}) = \frac{\partial \tilde{g}}{\partial g} \beta(g)$$ so for (*) to hold:

$$c \sum_{i=1}^{\infty} a_i g^i = \left(\sum_{J=1}^{\infty} a_J J g^{J-1} \right) \left(\sum_{k=1}^{\infty} b_k g^k \right)$$

$$= \sum_{J,k=1}^{\infty} a_J b_k J g^{J+k-1}$$

The double series on the right can be combined:

$$c \sum_{e=1}^{\infty} a_e g^e = \sum_{e=1}^{\infty} \sum_{J=1}^{e} a_J b_{e-J+1} J g^e$$

We get a system of linear equations for the unknowns a_i:

$$\sum_{J=1}^{e-1} a_J b_{e-J+1} J + a_e (b_1 e - c) = 0$$

$$e = 1, 2, 3, \ldots$$

This is a recurrence relation: a_e is fixed in terms of a_J, $J < e$. We only have to look at the initial condition:

$e = 1$ $$a_1 (b_1 - c) = 0$$

Either $c = b_1$, or $a_1 = 0$

We can choose a_i, so if we pick $a_1 \neq 0$, then $c = b_1$ and:

$$a_\ell = \frac{1}{b_1(\ell-1)} \sum_{j=1}^{\ell-1} a_j\, b_{\ell-j+1}\, j \qquad \ell > 1$$

determines all a_ℓ recursively if $b_1 \neq 0$, with a_1 arbitrary. This solves the exercise; we can find a coupling redefinition such that

$$\boxed{\tilde{\beta}(\tilde{g}) = b_1 \tilde{g}.}$$

- For completeness, consider choosing $a_1 = 0$. Then:

$$a_2(2b_1 - c) = 0$$

So again either $b_1 = \frac{c}{2}$ or $a_2 = 0$

Not all the a_i can be zero so, suppose the first $N-1$ are:

$$a_i = 0 \qquad i \leq N-1$$

$$a_N \neq 0$$

Then we have to choose $\quad c = b_1 N$

and we can still solve the recurrence relation.

However, in this second case $\tilde{\beta}(\tilde{g})$ is not meaningful. Indeed, close to the fixed point the coupling appears in the action as

$$\int d^d x \, g \mathcal{O} \approx \int d^d x \, \frac{\tilde{g}^{1/N}}{a_N^{1/N}} \mathcal{O}$$

so that the effective coupling is $\tilde{g}^{1/N}$ and:

$$\frac{d\tilde{g}^{1/N}}{d\log\Lambda} = \frac{1}{N} \tilde{g}^{\frac{1}{N}-1} \tilde{\beta}(\tilde{g}) = b_1 \tilde{g}^{1/N}$$

which is equivalent to the previous result.

The physical meaning of the redefinition is the following:

define an operator $\tilde{\mathcal{O}}$ as follows:

$$\tilde{\mathcal{O}} = \frac{g}{\tilde{g}} \mathcal{O} = \frac{1}{a_1}\left(1 - \frac{a_2}{a_1} g + \dots\right) \mathcal{O}$$

Then

$$\int d^d x \, g \mathcal{O} = \int d^d x \, \tilde{g} \tilde{\mathcal{O}}$$

And since $\frac{d\tilde{g}}{d\log\Lambda} = b_1 \tilde{g} \longrightarrow \tilde{g} = \tilde{g}_0 \Lambda^{b_1}$

$\tilde{\mathcal{O}}$ is a scaling operator and $\tilde{g}$ is its scaling variable.

b_1 is the eigenvalue of the RG transformation.

Exercise 1.3.5

$$f(h,t,\lambda) = \min_M \left[-hM + \frac{t}{2}M^2 + \frac{\lambda}{12}M^4 \right]$$

The minimum is unchanged if we redefine

$$M \to b^{\Delta} M \qquad \Delta \in \mathbb{R}$$

So the most general eigenvalues compatible with the scaling of the reduced free energy are determined by

$$\begin{aligned} f(b^{y_h}h, b^{y_t}t, b^{y_\lambda}\lambda) &= b^{y_h+\Delta} \min_M \left[-hM + b^{y_t - y_h + \Delta}\frac{t}{2}M^2 + \right. \\ &\quad \left. + b^{y_\lambda - y_h + 3\Delta}\frac{\lambda}{12}M^4 \right] \\ &= b^d f(h,t,\lambda) \end{aligned}$$

$$\Rightarrow \quad \begin{cases} y_h = d - \Delta \\ y_t = d - 2\Delta \\ y_\lambda = d - 4\Delta \end{cases}$$

The parameter α arises because we only considered a constant magnetization. Choosing $M = M(x)$ the additional kinetic term would further fix

$$\Delta = \frac{d}{2} - 1 \quad , \text{ so}$$

$$\begin{cases} y_h = \frac{d}{2} + 1 \\ y_t = 2 \\ y_\lambda = 4 - d \end{cases} \xrightarrow[\substack{\text{scaling argument} \\ \text{with 2 relevant} \\ \text{variables}}]{} \begin{cases} \alpha = 2 - \frac{d}{y_t} = 2 - \frac{d}{2} \\ \beta = \frac{d - y_h}{y_t} = \frac{d-2}{4} \\ \gamma = \frac{2y_h - d}{y_t} = 1 \\ \delta = \frac{y_h}{d - y_h} = \frac{d+2}{d-2} \end{cases}$$

The scaling argument that relates the eigenvalues to the critical exponents assumes that y_h and y_t are the only relevant eigenvalues. So, to begin with, it applies only when $d \geq 4$.

When $d>4$, though, it is further assumed that the free energy is well defined when $\lambda \to 0$ (u_I in the notation of the notes).
This is not the case, and so λ is called a dangerously irrelevant coupling.

To see this, consider the broken phase $t<0$:

$$\min_M \left(-hM - \frac{|t|}{2} M^2 + \frac{\lambda}{12} M^4 \right) \to -\infty \quad \text{as } \lambda \to 0$$

so the free energy does not have the scaling form used to derive the critical exponents at the Gaussian fixed point.

Exercise 1.2.6.

$$H = -\frac{1}{2} \sum_{x,y} \hat{J}(x-y)\, s(x)\, s(y) - \mathcal{H} \sum_x s(x) + \lambda \sum_x \left(s(x)^2 - 1 \right)^2$$

- As $\lambda \to \infty$ all configurations in which $s(x) \neq \pm 1$, $\forall x$, have infinite energy. Therefore they have zero Boltzman weight $e^{-\beta H}$ and do not contribute to the free energy.

The theory then becomes the Ising model

- As the lattice spacing goes to zero, we can approximate the Hamiltonian as follows:

$$\circledast \quad \sum_{x,y} \hat{J}(x-y)\, s(x)\, s(y) = \sum_{x} \sum_{\{\vec{e}\}} J \; s(x)\, s(x+a\vec{e})$$

where $\{\vec{e}\}$ is a orthonormal basis of $\mathbb{R}^d$
and $\hat{J}(x-y) = J$ if $y = x + a\vec{e}$

$$= - \sum_{x} \sum_{(\hat{e})} \frac{J}{2}\left[\left(s(x+a\vec{e}) - s(x)\right)^2 - s(x)^2 - s(x+a\hat{e})^2\right]$$

$$\underset{a\to 0}{\sim} - \frac{J}{2} \int \frac{d^d x}{a^d} \left[a^2 \frac{\partial s}{\partial x^i} \frac{\partial s}{\partial x^i} - \sum_{\{\hat{e}\}} \left(s(x)^2 + s(x+a\hat{e})^2\right)\right]$$

where i runs over all directions and Einstein summation convention is understood.

Define: $\quad \phi = \sqrt{J}\, a^{\frac{2-d}{2}}\, s \quad$ and:

$$\dots = - \int d^d x \left\{ \frac{1}{2} (\partial\phi)^2 - d a^{-2} \phi^2 \right\}$$

after a change of variables
$s(x+a\hat{e})^2 \to s(x)^2$

$$\otimes \quad \mathcal{H} \sum_x s(x) \underset{a\to 0}{\sim} \int d^d x \; a^{-\frac{d+2}{2}} \frac{\mathcal{H}}{\sqrt{J}} \phi$$

$$\otimes \quad \lambda \sum_x \left(s(x)^2 - 1\right)^2 \underset{a\to 0}{\sim} \int d^d x \; a^{-d} \lambda \left(\frac{a^{d-2}}{J} \phi^2 - 1 \right)^2$$

$$= \int d^d x \left(a^{d-4} \frac{\lambda}{J^2} \phi^4 - a^{-2} \frac{2\lambda}{J} \phi^2 + \text{const} \right)$$

All together:

$$H \underset{a\to 0}{\sim} \int d^d x \left[\frac{1}{2} (\partial\phi)^2 - a^{-2} \left(d + \frac{2\lambda}{J} \right) \phi^2 + \right.$$
$$\left. + \; a^{d-4} \frac{\lambda}{J^2} \phi^4 - a^{-\frac{d}{2}-1} \frac{\mathcal{H}}{\sqrt{J}} \phi + \dots \right]$$

It is useful to define:

$$\begin{cases} t = -d - \dfrac{2\lambda}{J} \\ u = \dfrac{\lambda}{J^2} \\ h = -\dfrac{\mathcal{H}}{\sqrt{J}} \end{cases}$$

We dropped a constant and terms with more than two derivatives.

We could also have started by more elaborated

lattice hamiltonians, with higher powers of $S(x)$.
The continuum limit would have been the same, up to terms of the general form:

$$g\, a^{-y_g} \text{"}\int d^d x\, \phi\, \partial^p \phi^{n-1}\text{"},$$

The only length scale is a, so $[\phi] = \frac{d-2}{2}$, $[\partial]=1$ and $[d^d x] = -d$ where $[\cdot]$ = mass dimension.

Since H is dimensionless,

$$y_g = d - \frac{d-2}{2} n - p$$

y_g is the RG eigenvalue of the coupling g at the Gaussian fixed point.

$y_g > 0$ only for small enough n and p.

$$y_g > 0 \Rightarrow \quad p < d - \frac{d-2}{2} n$$
and $p \geq 0$, $n \geq 1$

So: $1 \leq n < \frac{2d}{d-2} \underset{d>4}{\rightsquigarrow} 1 \leq n < 4$

$\left(\frac{2d}{d-2}\right.$ decreases$\left.\right)$

So for $d > 4$:

$y_g > 0$ for: $\bullet$ $n=1$ $p < \frac{d+2}{2}$

$p=0$: $\phi(x)$

$p>0$: only total derivatives

$\bullet$ $n=2$, $p<2$:

$p=0$: ϕ^2

for $p>0$: only total derivatives

$\bullet$ $n=3$ $p < 3 - \frac{d}{2}$

$\otimes$ $3 - \frac{d}{2} > 0 \to d < 6$

so when $4 < d < 6$ $(d=5)$

$p=0$: ϕ^3

But ϕ^3 is redundant as explained in the text.
So the only relevant couplings when $d>4$ are

h, t.

(But recall from previous exercise that u, there called λ, is dangerously irrelevant).